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Solutions of a Particle with Fractional δ-Potential in a Fractional Dimensional Space

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Abstract A Fourier transformation in a fractional dimensional space of order λ ($0 < \lambda \leq 1$) is defined to solve the Schrödinger equation with Riesz fractional derivatives of order α. This new method is applied for a particle in a fractional δ-potential well defined by $V(x) = -\gamma\delta^{\lambda}(x)$, where $\gamma > 0$ and $\delta^{\lambda}(x)$ is the fractional Dirac delta function. A complete solutions for the energy values and the wave functions are obtained in terms of the Fox H-functions. It is demonstrated that the eigen solutions are exist if $0 < \lambda < \alpha$. The results for $\lambda = 1$ and $\alpha = 2$ are in exact agreement with those presented in the standard quantum mechanics.

Keywords Fractional Delta function · Fox H-functions; fractional calculus · Fractional dimensional space

1 Introduction

In 1918 the Mathematician Felix Hausdorff introduced the notion of fractional dimension. This concept became very important especially after the revolutionary discovery of fractal geometry by Mandelbrot [1], where he used the concept of fractionality and worked out the relations between fractional dimension and integer dimension by using the scale method i.e. $d^{\lambda}x = \frac{\pi^{\lambda/2}|x|^{\lambda-1}}{\Gamma(\lambda/2)}dx$, $0 < \lambda \leq 1$. And numerous efforts has been made by researchers in various branches of science and technology [2–21]. Besides, there are other approaches to describe fractional dimension. These include, fractional calculus (a generalization of differentiation and integration to noninteger order) [22] and the analytic continuation of the dimension in Gaussian integral [13, 14, 23–25]. The later is often used in quantum field theory [24, 25], and introduced in the dimensional regularization method, $\int f(x)d^n x = \frac{2\pi^{(n)/2}}{\Gamma(\frac{n}{2})}\int_0^{\infty} f(x)x^{n-1}dx$ (a method of removing the divergent term in the evaluation of Feynman diagram term in order to avoid the loop divergence).

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Historically, the first example of fractional physical objects was the Brownian motion [1]. In quantum physics the first successful attempt applying of fractality concept was Feynman path integral approach [26], where Feynman and Hibbs [26] reformulated the non-relativistic quantum mechanics as a path integral over Brownian paths. Laskin [27, 28] used Lévy paths instead of the Brownian ones in the path integral and obtained a space fractional Schrödinger equation and developed the fractional quantum mechanics.

Related to fractal geometry and fractional dimensional space is the area of fractional derivatives and integrals which have recently been applied in many applications including particle physics [6], fractional Hamiltonian systems [7, 8], chaotic dynamics [4], astrophysics [9], physics of fractals and complex media [3] and recent studies of scaling phenomena [10–12]. Even though it was demonstrated in [3, 29], that the areas of fractals, fractional dimensional space and fractional derivatives are not completely independent, some authors have studied the Schrödinger equation which contains fractional derivative terms (Caputo or Riesz derivatives) [30, 31] in an integer space of order $n = 1, 2, \dots,$ and without taking into consideration, the fractionality dimensions of the fractals (irregularity or roughness). For example, Dong and Xu [30], solved the fractional Schrödinger equation using the quantum Riesz fractional operator introduced by Laskin [27, 28]. Naber [31] showed a time Caputo fractional Schrödinger equation. Wang and Xu [32] generalized the fractional Schrödinger equation to construct a space-time fractional Schrödinger equation.

Recently, Muslih and Agrawal [33] investigated the wave equation with Riesz fractional derivative and proved the connection between the Riesz derivative of order α and fractional space of order D as

$$(-\Delta)^{\alpha/2}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|^{D-\alpha}}\right) = \frac{2^{\alpha}\pi^{D/2}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{D-\alpha}{2})}\delta^{D}(\mathbf{r}-\mathbf{r}'), \tag{1}$$

where $\delta^D(\mathbf{r}-\mathbf{r}')$ is the D dimensional fractional Dirac delta function which satisfy the following identity

$$\int \delta^{D}(\mathbf{r}-\mathbf{r}')\, d^{D}r = 1. \tag{2}$$

Because of the fractional geometry nature of fractals, we will solve the fractional Schrödinger equation in fractional dimensional space. We will introduce the Fourier transform method in fractional dimensional space to solve this equation. This new method is applied to find a complete eigen solutions for a particle in a fractional δ-potential well defined by $V(x) = -\gamma\delta^{\lambda}(x)$, where $\gamma > 0$ and $\delta^{\lambda}(x)$ is the fractional Delta function.

Our paper is organized as follows: In Sect. 2, the Fourier transform method in fractional dimensional space is presented. In Sect. 3, we introduce Fourier transform method in fractional dimensional space to solve the fractional Schrödinger equation. Section 4 deals with the eigen solutions of a particle in fractional δ-potential well. Section 5 contains our conclusions.

2 Fourier Transform Method in Fractional Dimensional Space

The Fourier transform method in fractional dimensional space of Gaussian integral over fractional volume element dV_D is defined by Stillinger in Refs. [13, 14]. In this section, the Fourier transform $g(k)$ of a continues function $f(x)$ will be considered over fractional line element $d^\lambda x$, $0 < \lambda \le 1$ by using the Mandelbrot [1, 34] fractional line element which is defined as

$$d^\lambda x = \frac{\pi^{\lambda/2}|x|^{\lambda-1}}{\Gamma(\lambda/2)}dx, \tag{3}$$

where $0 < \lambda \le 1$. Then the Fourier transformation is defined as

$$g(k) = F(f(x)) = \int f(x)e^{ikx}d^\lambda x, \tag{4}$$

and the inverse Fourier transformation $f(x)$ is given by

$$f(x) = F^{-1}(g(k)) = \left(\frac{1}{2\pi}\right)^\lambda \int g(k)e^{-ikx}d^\lambda k. \tag{5}$$

The above definition can be generalized for N dimensional vector $\mathcal{R}^N$. The dimension of fractional space is given by $D = \lambda_1 + \lambda_2 + \cdots + \lambda_N$, $0 < \lambda_i \le 1$.

The *generalized Dirac delta function* in the λ dimensional fractional space is defined as

$$\delta^\lambda(x - x') = \left(\frac{1}{2\pi}\right)^\lambda \int e^{ik(x-x')}d^\lambda k. \tag{6}$$

Theorem 1 *The generalized Dirac delta function in the λ dimensional fractional space satisfies the following identity*

$$\delta^\lambda(x) = \lim_{\varepsilon\to\infty} \varepsilon^\lambda e^{-\pi\varepsilon^2 x^2}. \tag{7}$$

Proof From (6), we have

$$\int_{-\infty}^{\infty} e^{ikx}d^\lambda k = (2\pi)^\lambda \delta^\lambda(x). \tag{8}$$

Now, let us define the following integral [13, 14, 23, 35]

$$I(\beta, q) = \int_{-\infty}^{\infty} e^{(-\beta x^2 + qx)}d^\lambda x = \beta^{-\lambda/2}\pi^{\lambda/2}e^{q^2/4\beta}. \tag{9}$$

Let $q = ik$ and $x \to k$, this yields

$$\int_{-\infty}^{\infty} e^{(-\beta k^2 + ikx)}d^\lambda k = \beta^{-\lambda/2}\pi^{\lambda/2}e^{-x^2/4\beta}. \tag{10}$$

Hence, we arrive to the value of Dirac delta function as

$$\begin{aligned}\delta^\lambda(x) &= \left(\frac{1}{2\pi}\right)^\lambda \int_{-\infty}^{\infty} e^{ikx}d^\lambda k = \left(\frac{1}{2\pi}\right)^\lambda \lim_{\beta\to 0}\left(\int_{-\infty}^{\infty} e^{(-\beta k^2 + ikx)}d^\lambda k\right) \\ &= \left(\frac{1}{2\pi}\right)^\lambda \lim_{\beta\to 0}\left(\beta^{-\lambda/2}\pi^{\lambda/2}e^{-x^2/4\beta}\right).\end{aligned} \tag{11}$$

Let $\beta = \frac{1}{4\pi\varepsilon^2}$, we arrive at the proof of Theorem 1. □

Theorem 2 *The generalized Dirac delta function, satisfy the following identities*

(1)
$$\int_{-\infty}^{\infty} f(x)\delta^{\lambda}(x)d^{\lambda}x = f(0). \tag{12}$$

(2)
$$\int_{-\infty}^{\infty} \delta^{\lambda}(x)d^{\lambda}x = 1. \tag{13}$$

Proof Using the scaling method of Mandelbrot (see (3)), we have

$$\int_{-\infty}^{\infty} \delta^{\lambda}(x)f(x)d^{\lambda}x = \frac{2\pi^{\lambda/2}}{\Gamma(\lambda/2)}\left(\int_{0}^{\infty} \delta^{\lambda}(x)f(x)x^{\lambda-1}dx\right). \tag{14}$$

Substitution of Dirac delta function defined in (7), we obtain

$$\int_{-\infty}^{\infty} \delta^{\lambda}(x)f(x)d^{\lambda}x = \frac{2\pi^{\lambda/2}}{\Gamma(\lambda/2)}\left(\int_{0}^{\infty} \lim_{\varepsilon\to\infty} \varepsilon^{\lambda}e^{-\pi\varepsilon^2x^2}f(x)x^{\lambda-1}dx\right). \tag{15}$$

Let $z=\pi\varepsilon^2x^2$, then (15) can be written as

$$\int_{-\infty}^{\infty} \delta^{\lambda}(x)f(x)d^{\alpha}x = \frac{1}{\Gamma(\lambda/2)}\left(\lim_{\varepsilon\to\infty} f(0)\int_{0}^{\infty} e^{-z}z^{(\lambda/2-1)}dz\right) = f(0). \tag{16}$$

In particular, when $f(x)=1$, we arrive at

$$\int_{-\infty}^{\infty} \delta^{\lambda}(x)d^{\lambda}x = 1. \tag{17}$$

In general the above definitions can be generalized for N dimensional vector $\mathcal{R}^N$. The dimension of fractional space is given by $D=\lambda_1+\lambda_2+\cdots+\lambda_N$, $0<\lambda_i\le 1$. In this case the generalized Dirac delta function in the D dimensional fractional space satisfies the following identity

$$\begin{aligned}\delta^{D}(\mathbf{r}-\mathbf{r}') &= \left(\frac{1}{2\pi}\right)^{D}\int e^{\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}d^{D}k \\ &= \delta^{\lambda_1}(x_1-x_1')\delta^{\lambda_2}(x_2-x_2')\cdots\delta^{\lambda_N}(x_N-x_N'). \end{aligned}\tag{18}$$

The Fourier convolution operator of two functions h and φ is defined by the integral

$$h*\varphi := (h*\varphi)(x) = \int_{-\infty}^{\infty} h(x-y)\varphi(y)d^{\lambda}y \quad (x\in R). \tag{19}$$

The Fourier transforms in spatial and momentum space can be obtained after the changing of variables from k to the momentum $p=\hbar k$ as

$$\phi(x) = \left(\frac{1}{2\pi\hbar}\right)^{\lambda}\int_{-\infty}^{\infty} \varphi(p)e^{-ipx/\hbar}d^{\lambda}p, \tag{20}$$

$$\varphi(p) = \int_{-\infty}^{\infty} \phi(p)e^{ipx/\hbar}d^{\lambda}x. \tag{21}$$

□

3 Fractional Schrödinger Equation in Fractional Space

In this section, we will investigate the fractional Schrödinger equation in fractional space. Laskin [27, 28], developed the fractional Feynman path integral over Levy paths using the Hamiltonian which includes the fractional kinetic term in terms of the quantum Riesz fractional operator and is given by $(-\hbar^2\Delta)^{\alpha/2}$, $1 < \alpha \leq 2$, where $\Delta = \frac{\partial}{\partial x^2}$ is the Laplacian in one dimension. The time dependent Schrödinger equation is expressed as

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = H_\alpha\psi(x,t), \tag{22}$$

where H_α is the Laskin fractional Hamiltonian and defined as

$$H_\alpha = D_\alpha(-\hbar^2\Delta)^{\alpha/2} + V(x,t). \tag{23}$$

Here, D_α has physical dimension $[D_a] = erg^{1-\alpha} \times cm^\alpha \times s^{-\alpha}$ ($D_\alpha = \frac{1}{2m}$ for $\alpha = 2$, m is the physical mass of the particle).

For Hamiltonians do not depend explicitly on time and taking into account that $\psi(x,t) = \phi(x)e^{-iE/\hbar}$, then (22) can be put in the form.

$$D_a(-\hbar^2\Delta)^{\alpha/2}\phi(x) + V(x)\phi(x) = E\phi(x). \tag{24}$$

As was specified in the introduction, the solutions of time dependent and independent fractional Schrödinger equations in the integer dimensional space, are studied by some authors [30–32]. Our aim is to use the method introduced in the previous section and to investigate the eigen solutions of fractional Schrödinger in fractional dimensional space. As an example we will solve a fractal source with a potential in the form of fractional Dirac delta distribution function. To accomplish this goal, we multiply (24) from left by $e^{-ipx/\hbar}$ and taking the Fourier transform of the resultant value, and then integrating over x from $-\infty$ to ∞ we have,

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left(D_\alpha|p'|^\alpha\varphi(p')d^\lambda p'\left\{\int e^{ix/h(p'-p)}d^\lambda x\right\}\right) \\
&\quad + \int_{-\infty}^{\infty} e^{-ix/h(p-p')}V(x)d^\lambda x\varphi(p')d^\lambda p' \\
&= E\int_{-\infty}^{\infty} e^{-ix/h(p-p')}d^\beta x\varphi(p')d^\lambda p'.
\end{aligned} \tag{25}$$

Using the definition of fractional Dirac delta function (6), we obtain

$$\begin{aligned}
&(2\pi\hbar)^\lambda(D_\alpha|p|^\alpha\varphi(p)) + \int_{-\infty}^{\infty} e^{-ix/h(p-p')}V(x)d^\lambda x\varphi(p')d^\lambda p' \\
&= (2\pi\hbar)^\lambda E\varphi(p).
\end{aligned} \tag{26}$$

4 Fractional δ-Potential

In this section, we consider a particle in fractional δ-potential well defined by $V(x) = -\gamma\delta^\lambda(x)$, $\gamma > 0$ and $0 < \lambda \leq 1$, where $\delta^\lambda(x)$ is the fractional Dirac delta function defined in (6). The time independent fractional Schrödinger equation for this particle is given by

$$D_a(-\hbar^2\Delta)^{\alpha/2}\phi(x) - \gamma\delta^\lambda(x)\phi(x) = E\phi(x). \tag{27}$$

We consider here $E < 0$. Making use of (26), we obtain

$$(2\pi\hbar)^\lambda(D_\alpha|p|^\alpha\varphi(p)) + \int_{-\infty}^{\infty} e^{-ix/h(p-p')}(-\gamma\delta^\lambda(x))d^\lambda x\varphi(p')d^\lambda p'$$
$$= (2\pi\hbar)^\lambda E\varphi(p). \tag{28}$$

Using the property (6), we obtain

$$(D_\alpha|p|^\alpha\varphi(p)) - \frac{\gamma}{(2\pi\hbar)^\lambda}\int_{-\infty}^{\infty}\varphi(p)d^\lambda p = E\varphi(p). \tag{29}$$

Let

$$\int_{-\infty}^{\infty}\varphi(p)d^\lambda p = C, \tag{30}$$

where C is a constant. Substituting (30) in (29), we have

$$\varphi(p) = \frac{-\gamma}{(2\pi\hbar)^\lambda}\frac{C}{D_\alpha|p|^\alpha - E}. \tag{31}$$

Again, substitution of the solution (31) in (30), we obtain

$$\int_{-\infty}^{\infty}\frac{d^\lambda p}{D_\alpha|p|^\alpha - E} = \frac{(2\pi\hbar)^\lambda}{\gamma}. \tag{32}$$

Making use of the fractional line element defined in (3), then (32) can be put in the form

$$\frac{2\pi^\lambda/2}{\Gamma(\lambda/2)}\int_0^{\infty}\frac{|p|^{\lambda-1}dp}{D_\alpha|p|^\alpha - E} = \frac{(2\pi\hbar)^\lambda}{\gamma}. \tag{33}$$

Using the identity [36]

$$\frac{z^\lambda}{1+az^\alpha} = a^{\lambda/\alpha}H_{1,1}^{1,1}\left[az^\alpha\Big|_{(\lambda/\alpha,1)}^{(\lambda/\alpha,1)}\right], \tag{34}$$

where $H_{p,q}^{m,n}(z)$ is the H Fox-function (for more details, see the Appendix), we obtain

$$\frac{|p|^{\lambda-1}}{D_\alpha|p|^\alpha - E} = -\frac{1}{E}\left(\frac{D_\alpha}{-E}\right)^{-(\frac{\lambda-1}{\alpha})}H_{1,1}^{1,1}\left[\left(\frac{D_\alpha}{-E}\right)|p|^\alpha\Bigg|_{((\lambda-1)/\alpha,1)}^{((\lambda-1)/\alpha,1)}\right]. \tag{35}$$

This allows us to calculate the integral

$$\int_0^{\infty}\frac{|p|^{\lambda-1}dp}{D_\alpha|p|^\alpha - E} = -\frac{1}{\alpha E}\left(-\frac{E}{D_\alpha}\right)^{(\frac{\lambda-2}{\alpha})}\int_0^{\infty}H_{1,1}^{1,1}\left[\left(\frac{D_\alpha}{-E}\right)^{1/\alpha}|p|^\alpha\Bigg|_{((\lambda-1)/\alpha,1/\alpha)}^{((\lambda-1)/\alpha,1/\alpha)}\right]dp. \tag{36}$$

Using the identity (49), we obtain the eigen values $E_{\alpha,\lambda}$ for the particle in fractional δ-potential well as

$$E_{\alpha,\lambda} = -\left(\frac{\gamma\Gamma(\lambda/\alpha)\Gamma(1-\lambda/\alpha)}{2^{\lambda-1}\hbar^\lambda\pi^\lambda\Gamma(\lambda/2)\alpha(D_a)^{\lambda/\alpha}}\right)^{\frac{\alpha-\lambda}{\alpha}}. \tag{37}$$

According to the identity (49), the integral (36) exists, when $0 < \lambda < \alpha$. For one dimensional fractal systems $0 < \lambda \leq 1$ we obtain $1 < \alpha$. As special cases, for $\lambda = 1, \alpha = 2$, the value of $E_{\alpha,\lambda}$ reduces to the same energy eigen value as given in standard mechanics [37, 38].

Making use of (36) and (20), we obtain the solution $\phi(x)$ as

$$\begin{aligned}\phi(x) &= \frac{\gamma C}{(2\pi\hbar)^{2\lambda}} \int_{-\infty}^{\infty} \frac{e^{ipx/\hbar}}{D_\alpha |p|^\alpha - E} d^\lambda p \\ &= \frac{\gamma C}{(2\pi\hbar)^{2\lambda}} \frac{2\pi^\lambda/2}{\Gamma(\lambda/2)} \int_0^{\infty} \frac{e^{ipx/\hbar}|p|^{\lambda-1}}{D_\alpha |p|^\alpha - E} dp \\ &= -\frac{\gamma C}{(2\pi\hbar)^{2\lambda}} \frac{2\pi^\lambda/2}{\Gamma(\lambda/2)E} \int_0^{\infty} \frac{e^{ipx/\hbar}|p|^{\lambda-1}}{1 - D_\alpha/E|p|^\alpha} dp. \end{aligned} \tag{38}$$

Again, using the identity formula (34), we obtain

$$\begin{aligned}\phi(x) &= -\frac{\gamma C}{(2\pi\hbar)^{2\lambda}} \frac{2\pi^\lambda/2}{\Gamma(\lambda/2)E} \\ &\quad \times \int_0^{\infty} e^{ipx/\hbar} \left(\frac{D_\alpha}{-E}\right)^{-(\frac{\lambda-1}{\alpha})} H_{1,1}^{1,1}\left[\left(\frac{D_\alpha}{-E}\right)|p|^\alpha \middle|_{((\lambda-1)/\alpha,1)}^{((\lambda-1)/\alpha,1)}\right] dp. \end{aligned} \tag{39}$$

The integral (39) can be evaluated using the Fourier cosine transform of the H function [39] as follows:

$$\phi(x) = C_\alpha^\lambda F_\alpha^\lambda(x), \tag{40}$$

where C_α^λ and $F_\alpha^\lambda(x)$ are given respectively as

$$C_\alpha^\lambda = \frac{\gamma C}{(2\pi\hbar)^{2\lambda}} \frac{2\pi^\lambda/2}{\Gamma(\lambda/2)} \frac{1}{(D_\alpha)^{\frac{\lambda-1}{\alpha}}(-E)^{\frac{\alpha+1-\lambda}{\alpha}}}, \tag{41}$$

$$F_\alpha^\lambda(x) = \int_0^{\infty} \cos(px/\hbar) H_{1,1}^{1,1}\left[\left(\frac{D_\alpha}{-E}\right)|p|^\alpha \middle|_{((\lambda-1)/\alpha,1)}^{((\lambda-1)/\alpha,1)}\right] dp. \tag{42}$$

With the help of (52), $F_\alpha^\lambda(x)$ can be evaluated as

$$F_\alpha^\lambda(x) = \frac{\pi\hbar}{|x|} H_{2,3}^{2,1}\left[\left(\frac{|x|}{\hbar}\right)^\alpha \left(\frac{-E}{D_\alpha}\right) \middle|_{(1,\alpha/2),(\frac{\alpha-\lambda+1}{\alpha},1),(1,\alpha/2)}^{(\frac{\alpha-\lambda+1}{\alpha},1),(1,\alpha/2)}\right]. \tag{43}$$

Using (53), (43), reduces to

$$F_\alpha^\lambda(x) = \frac{\pi\hbar}{\alpha} H_{2,3}^{2,1}\left[|x|\left(\frac{D_\alpha \hbar^\alpha}{-E}\right)^{-1/\alpha} \middle|_{(0,1),(\frac{\alpha-\lambda}{\alpha},1/\alpha),(1/2,1/2)}^{(\frac{\alpha-\lambda}{\alpha},1/\alpha),(1/2,1/2)}\right]. \tag{44}$$

Furthermore, using the formulas (54) and (55), we obtain

$$F_\alpha^\lambda(x) = \frac{\pi\hbar}{\alpha} H_{0,1}^{1,0}\left[|x|\left(\frac{D_\alpha \hbar^\alpha}{-E}\right)^{-1/\alpha} \middle| (0,1)\right]. \tag{45}$$

Finally we obtain the wave function $\phi(x)$ for a particle in fractional δ-potential well as

$$\phi(x) = C_\alpha^\lambda \frac{\pi\hbar}{\alpha} H_{0,1}^{1,0}\left[|x|\left(\frac{D_\alpha \hbar^\alpha}{-E}\right)^{-1/\alpha}\Big|(0,1)\right]. \tag{46}$$

For the special case $\alpha = 2$ and $\lambda = 1$, the wave function in (46), reduces to that in the standard quantum mechanics [37, 38].

5 Conclusions

In this paper we developed the Fourier transform in fractional space. This allows us solve fractional Schrödinger in fractional dimensional space λ. As an example we obtained a complete eigen solutions for a particle in a fractional δ-potential in the form $V(x) = -\gamma\delta^\lambda(x)$. where $\gamma > 0$ and $\delta^\lambda(x)$ is the fractional Dirac delta function. Using the properties of Fox H-functions we demonstrated that the eigen solutions are exist if $0 < \lambda < \alpha$.

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Appendix

In this appendix, we will review the Fox's H-function, and its properties, which have been used in our calculations. The Fox's H-function is defined by the contour integral, [36]

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = \frac{1}{2\pi i}\int_L h(s) z^s\, ds, \tag{47}$$

where $h(s)$ is given by

$$h(s) = \frac{\prod_{j=1}^m \Gamma(b_j + B_j s)\prod_{j=1}^n \Gamma(1 - a_j + A_j s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + B_j s)\prod_{j=n+1}^p \Gamma(a_j + A_j s)}. \tag{48}$$

Where m, n, p, q are integers satisfying $m^2 + n^2 \neq 0$, $0 \le n \le p$, $0 \le m \le q$ and empty products are interpreted as unity. The parameters a_j $(j = 1, \dots, p)$ and $b_j(1, \dots, q)$ are complex numbers and A_j $(j = 1, \dots, p)$ and B_j $(j = 1, \dots, q)$ are positive numbers satisfying $P_a \cap P_b = \emptyset$, where $P_a = \{s = (b_j + k)/B_j, j = 1, 2, \dots, m; k = 0, 1, 2, \dots\}$ and $P_b = \{s = (A_j - 1 - k)/A_j, j = 1, 2, \dots, m; k = 0, 1, 2, \dots\}$. The integration contour runs from $s = c - i\infty$ to $s = c + i\infty$ such that P_a lies to right of L and P_b to the left of L. The Mellin transform of a single H-function is given as

$$\int_0^\infty z^{s-1} H_{p,q}^{m,n}\left[az\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] dz$$
$$= a^{-s}\frac{\prod_{j=1}^m \Gamma(b_j + B_j s)\prod_{j=1}^n \Gamma(1 - a_j + A_j s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + B_j s)\prod_{j=n+1}^p \Gamma(a_j + A_j s)}, \tag{49}$$

where $-\min_{1\le j\le m} R(\frac{bj}{B_j}) < R(s) < \frac{1}{A_j} - \max_{1\le j\le n} R(\frac{aj}{A_j})$, $|\arg a| < \frac{1}{2}\pi\lambda$, $\lambda = \sum_{j=1}^m A_j - \sum_{j=n+1}^p A_j \sum_{j=1}^m B_j - \sum_{j=m+1}^q B_j > 0$.

Now we would like to derive the cosine transform of H-function. Using the integral [36]

$$\int_0^\infty z^{s+1/2} J_\nu(kz) H_{p,q}^{m,n}\left[az^\mu\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]dz$$
$$= 2^{s+1/2}k^{-s-3/2} H_{p+2,q}^{m,n+1}\left[(2/k)^\mu z\Big|_{(b_q,B_q)}^{(1/4-s/2-\nu/2,\mu/2),(a_p,A_p),(1/4-s/2+\nu/2,\mu/2)}\right], \quad (50)$$

and the values of cosine function

$$J_{-1/2}(kz) = \sqrt{\frac{2}{\pi kz}}\cos kz, \quad (51)$$

we obtain the cosine transform of H-function as

$$\int_0^\infty z^{s-1}\cos(kz) H_{p,q}^{m,n}\left[az^\mu\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]dz = \frac{\pi}{ks} H_{q+1,p+2}^{m+1,n}\left[k^\mu/a\Big|_{(s,\mu),(1-aq,Aq),(\frac{1+s}{2},\mu/2)}^{(1-bq,Bq),(\frac{1+s}{2},\mu/2)}\right]. \quad (52)$$

Some properties of H-function

$$H_{p,q}^{m,n}\left[z^\mu\Big|_{(b_q,B_q)}^{(a_p,A_p)}\right] = \frac{1}{\mu} H_{p,q}^{m,n}\left[z\Big|_{(b_q,\frac{B_q}{\mu})}^{(a_p,\frac{A_p}{\mu})}\right], \quad \mu > 0. \quad (53)$$

$$H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),(b_2,B_2),\ldots,(b_{q-1},B_{q-1},(a_1,A_1))}^{(a_1,A_1),(a_2,A_2),\ldots,(a_p,A_p)}\right] = H_{p-1,q-1}^{m,n-1}\left[z\Big|_{(b_1,B_1),(b_2,B_2),\ldots,(b_{q-1},B_{q-1})}^{(a_2,A_2),\ldots,(a_p,A_p)}\right]. \quad (54)$$

$$H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),(b_2,B_2),\ldots,(b_q,B_q)}^{(a_1,A_1),(a_2,A_2),\ldots,(b_1,B_1)}\right] = H_{p-1,q-1}^{m-1,n}\left[z\Big|_{(b_2,B_2),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_{p-1},A_{p-1})}\right]. \quad (55)$$

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